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LETTER TO THE EDITOR

The structure of gauged internal symmetry groups

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Abstract. It is shown that under the usual assumptions with regard to the Lagrangian the most general case of a gauged internal symmetry group is given by $G = G_1 \times \ldots G_m$, where the G_i factors are either $(\mathbf{R}, +)$ or $U(1)$ or compact simply connected simple Lie groups. The special circumstances under which $(\mathbf{R}, +)$ appears are pointed out.

In particle physics, definitions of a gauge theory always take for granted that the gauge group G is compact. To be more precise, it is assumed to be either $U(1)$ or a simple compact group or a finite number of products of such groups. As a justification for this assumption, one usually quotes the theorem that non-compact Lie groups, and therefore their Lie algebras as well, have no finite-dimensional unitary representations apart from the trivial one [1]. This argument is readily extended to semisimple Lie groups: as their Lie algebras are direct sums of simple Lie algebras [2, 3] any possible non-compact summand is represented trivially if one admits only a finite number of fields transforming unitarily under G in the Lagrangian. Therefore, the non-compact factors leave no trace in the theory. However, this argument is not quite conclusive because non-semisimple Lie groups permit unitary finite-dimensional representations in general. A typical example is $(\mathbf{R}, +)$, the additive group of real numbers which has continuous one-dimensional irreducible representations $x \to \exp \lambda x$ which are unitary for imaginary λ and non-unitary otherwise.

The group $G = (\mathbf{R}, +)$ has yet another instructive feature. Namely, it possesses unitary representations D such that $G/\ker D$ is not compact. A generic example is

$$D: \quad x \to \begin{pmatrix} e^{2\pi i x} & 0 \\ 0 & e^{2\pi i \mu x} \end{pmatrix}$$

where $\ker D = \{0\}$ (μ is irrational). Thus factorizing out the kernel of a unitary representation does not automatically lead to a compact group and we seem to be forced to admit also non-compact gauge groups. However, we will see that the only possible non-compact factors in $G/\ker D$ with unitary D are those isomorphic to $(\mathbf{R}, +)$. At the end of this letter we will also discuss the special circumstances under which such a factor appears in the gauge group.

We are aware that in principle the 'compactness' of the internal symmetry group and some points made in this letter are already contained in [4]. Nevertheless, we think it is of interest to work out in detail the arguments leading to the structure of the gauge group independently of the complications introduced by representations of the Poincaré group and the properties of the S matrix. In other words, we want to discuss this problem as a clearly defined instructive exercise in Lie groups and algebras.

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In the following, the internal symmetry group will be a connected Lie group since all connected components of a Lie group can be reached from the component of the unit element by discrete transformations which have no bearing on the gauge structure of the theory. Actually, contact between the gauge principle and the group is only made by the requirement that the group be a Lie group because the Lie algebra is necessary for the formulation of the gauge couplings. Now we will show that under the usual assumptions for gauge theories, *the gauge group can always be taken compact apart from possible* $(\mathbf{R}, +)$ factors.

Theorem. Basing the construction of a gauge Lagrangian on the requirement that in the theory only a finite number of fields appear such that the fields constitute the vector space of a finite-dimensional unitary representation of the gauge group, then one can always choose the gauge group to be of the type $G = G_1 \times \ldots \times G_m$, where the G_l factors are either $(\mathbf{R}, +)$ or $U(1)$ or compact simply connected simple Lie groups.

To be as clear as possible, the proof will be divided into a series of steps.

(1) We start with a connected Lie group G' and assume that the fields in the Lagrangian are in a finite-dimensional unitary representation D on a complex vector space V with $\dim V = n$. Then $\ker D$ is a closed subgroup of G' and it is sufficient to consider $G_0 = G'/\mathrm{ker}D$ as a gauge group since $\mathrm{ker}D$ has no effect on the Lagrangian. Then G_0 is again a Lie group [5] and we can consider D as a representation of G_0. Furthermore, G_0 is diffeomorphic to $D(G_0)$ [5] which is a subgroup of $GL(V)$, the group of linear transformations on V.

(2) Now we pass over to the real Lie algebras $\mathcal{L}$ of G_0 and $\mathcal{L}_D$ of $D(G_0)$ which are isomorphic. $\mathcal{L}_D$ is a representation of $\mathcal{L}$ which, for simplicity, we again denote by D. Since D is unitary, the elements X of $\mathcal{L}_D$ are anti-Hermitian operators, i. e. $X^\dagger = -X$, and we can decompose $D = D_1 \oplus \ldots \oplus D_k$ acting on $V = V_1 \oplus \ldots \oplus V_k$ into irreducible representations D_α acting on V_α.

(3) After these preliminaries, we want to consider the radical $\mathcal{R}_D$ of $\mathcal{L}_D$, i. e. the largest solvable ideal of $\mathcal{L}_D$. To do this we are motivated by the theorem that the quotient of a Lie algebra by its radical is semisimple [2, 3]. First we will show that the unitarity of $\mathcal{L}_D$ entails that $\mathcal{R}_D$ is Abelian.

Since $\mathcal{R}_D$ is a solvable Lie algebra acting on the vector space V we can apply Lie's theorem [2, 3] which tells us that there is a joint non-zero eigenvector v_1 of all $Y \in \mathcal{R}_D$. Because of $Y^\dagger = -Y$ the space $((v_1))^\perp$, i. e. the orthogonal complement of the linear span of v_1, is invariant under $\mathcal{R}_D$. Thus we can apply Lie's theorem once more and find a second eigenvector $v_2 \perp v_1$. Iterating this process, a common orthogonal basis of joint eigenvectors of $\mathcal{R}_D$ is constructed which means that $\mathcal{R}_D$ is Abelian.

(4) Now we will show that all the elements Y of $\mathcal{R}_D$ are proportional to the unit operator when restricted to any of the $V_\alpha (\alpha = 1, \ldots, k)$, i. e.

$$Y|_{V_\alpha} \propto \mathbf{1}_\alpha. \tag{1}$$

To do this, let us fix an element $Y \in \mathcal{R}_D$ and a space V_α. Then V_α can be decomposed into eigenspaces W_β of Y, i. e. $Yx = \lambda_\beta x$ for all $x \in W_\beta$ and $\lambda_\beta \neq \lambda_{\beta'}$ and $W_\beta \perp W_{\beta'}$ for $\beta \neq \beta'$. Since $\mathcal{R}_D$ is Abelian, all the spaces W_β are invariant under $\mathcal{R}_D$ and since it is an ideal, we have

$$[Y, X] = Y' \in \mathcal{R}_D \tag{2}$$

for all $X \in \mathcal{L}_D$. Now taking $w_\beta \in W_\beta$ and $w_\gamma \in W_\gamma (\beta \neq \gamma)$ we obtain

$$\begin{aligned}\lambda_\beta \langle w_\beta | X w_\gamma \rangle &= \langle -Y w_\beta | X w_\gamma \rangle = \langle w_\beta | Y X w_\gamma \rangle \\ &= \langle w_\beta | (XY + Y') w_\gamma \rangle = \langle w_\beta | \lambda_\gamma X w_\gamma + Y' w_\gamma \rangle = \lambda_\gamma \langle w_\beta | X w_\gamma \rangle\end{aligned}$$

where $Y'W_\gamma \subseteq W_\gamma$ was taken into account. (Note also that the eigenvalues of Y are imaginary.) We see therefore that all W_β are invariant under $\mathcal{L}_D$. Since D_α is irreducible, there can only be a single space W_β being the entire V_α. This proves the above statement.

(5) We have seen in (4) that the radical $\mathcal{R}_D$ is identical with the centre of $\mathcal{L}_D$. This allows the decomposition

$$\mathcal{L}_D = \mathcal{L}'_D \oplus \mathcal{R}_D \tag{3}$$

where $\mathcal{L}'_D$ is semisimple and consists of all elements $X \in \mathcal{L}_D$ with $\mathrm{Tr}_\alpha X = 0$ for all $\alpha = 1, \ldots, k$ (Tr_α denotes the trace being taken in the subspace V_α of V). Consequently, $\mathcal{L}'_D$ is a Lie subalgebra of $su(n)$ and therefore compact, i.e. its Killing form is negative definite.

To prove (3), we use that $\mathcal{L}_D/\mathcal{R}_D$ is semisimple as already mentioned in (3). A semisimple real or complex Lie algebra $\mathcal{A}$ is identical to its derived Lie algebra $\mathcal{A}' = [\mathcal{A}, \mathcal{A}]$. (This can be seen e.g. in the following way: for semisimple Lie algebras the Killing metric is a non-singular matrix which can be used to invert the commutator relations of the basis elements leading to an expression of the basis elements as linear combinations of commutators.) In other words, in a semisimple Lie algebra *every* element can be written as a linear combination of Lie products. Applying this to $X + \mathcal{R}_D \in \mathcal{L}_D/\mathcal{R}_D$ one obtains

$$X + \mathcal{R}_D = \sum_i [X'_i + \mathcal{R}_D, X'_i + \mathcal{R}_D] = \sum_i [X'_i, X'_i] + \mathcal{R}_D$$

or

$$X = Z + Y \quad \text{with} \quad \mathrm{Tr}_\alpha Z = 0 \ (\alpha = 1, \ldots, k) \quad \text{and} \quad Y \in \mathcal{R}_D. \tag{4}$$

(6) Since $\mathcal{L}$ is isomorphic to $\mathcal{L}_D$ equation (3) is also valid for $\mathcal{L}$, i.e. $\mathcal{L} = \mathcal{L}' \oplus \mathcal{R}$ with $\mathcal{L}'$ semisimple and compact and $\mathcal{R}$ being the Abelian radical of $\mathcal{L}$. Since $\mathcal{L}' = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_s$ is a direct sum of simple Lie algebras [1–3], we conclude that the universal covering group $\widetilde{G}_0$ of G_0 is given by [1]

$$\widetilde{G}_0 = (\mathbf{R}, +) \times \ldots \times (\mathbf{R}, +) \times \widetilde{G}_1 \times \ldots \times \widetilde{G}_s \tag{5}$$

with $\widetilde{G}_j$ being simply connected simple Lie groups. There are as many factors $\mathbf{R}$ in $\widetilde{G}_0$ as the dimension of $\mathcal{R}$. Since a semisimple group is compact if and only if its Lie algebra is compact [1], the groups $\widetilde{G}_j$ ($j = 1, \ldots, s$) are compact.

Now we have practically completed the proof. It only remains to discuss when $(\mathbf{R}, +)$ can be curled up to $U(1)$. Since we are considering gauge theories, there is an independent gauge coupling constant for every factor in $\widetilde{G}_0$ in equation (5). (The coupling constants are independent unless there are additional symmetries connecting equal factors in $\widetilde{G}_0$.) Furthermore, for every $\mathcal{L}_j$ ($j = 1, \ldots, s$) one has to choose an orthonormal basis of generators $\{X_a\}$ in $\mathcal{L}_D$, i. e. $\mathrm{Tr}(X_a X_b) = -\delta_{ab}$, and write down the usual gauge couplings. In $\mathcal{R}_D$ we have to take a basis of linearly independent generators Y_a ($a = 1, \ldots, r$) each associated with a gauge field A_a and a coupling constant g_a. From equation (1) and from the form of $\mathcal{L}'_D$ (equation (3)) it is clear that $\mathrm{Tr}(X_a Y_b) = 0$ for all a, b. The generators of $\mathcal{R}_D$ do not *a priori* form an orthonormal set. However, by an orthogonal rotation R on the Abelian gauge fields $A_a = \sum_b R_{ab} A'_b$ one can obtain such a set of generators by

$$\sum_a g_a Y_a A_a = \sum_b g'_b Y'_b A'_b$$

with

$$g'_b Y'_b = \sum_a g_a Y_a R_{ab}. \tag{6}$$

The requirement

$$\mathrm{Tr}(Y'_a Y'_b) = -\delta_{ab} \tag{7}$$

then leads to

$$R^T \mathcal{M} R = -\mathrm{diag}(g'^2_1, \ldots, g'^2_r)$$

with

$$\mathcal{M}_{ab} = g_a g_b \,\mathrm{Tr}(Y_a Y_b). \tag{8}$$

The $r \times r$ matrix $\mathcal{M}$ is a symmetric negative definite matrix ($\det \mathcal{M} \neq 0$ since the generators are linearly independent) and can therefore be diagonalized by the rotation matrix R giving the new coupling constants g'_a. Thus one can always choose an orthonormal set of generators for the entire Lie algebra $\mathcal{L}_D$.

In constructing the gauge Lagrangian, the generators $Y'_a \in \mathcal{R}_\mathcal{D}$ are diagonal matrices. If the matrix Y'_a/Y'_{a11} has only rational elements then one can consider $x \to \exp(xY'_a)$ as a representation of $U(1) \cong (\mathbf{R}, +)/u\mathbf{Z}$ with a suitable number $u \in \mathbf{R}$. If this is not the case we have to keep the non-compact factor $(\mathbf{R}, +)$. Possible factors of this type are the only source of non-compactness in the gauge group G. Thus, the difference between $\widetilde{G}_0$ and G of the theorem is only given by the replacement of $(\mathbf{R}, +)$ by $U(1)$ wherever, this is possible.

Let us dicuss in more detail the case where not all ratios of matrix elements in Y'_a are rational. For simplicity we assume that there are only two sets of incommensurable matrix elements. Thus Y'_a is given by

$$Y'_a = \mathrm{i}(u_1 T_1 + u_2 T_2) \tag{9}$$

with $u_1, u_2 \in \mathbf{R}$, u_1/u_2 being irrational and diagonal matrices T_1, T_2 having only rational entries. Since a gauge Lagrangian is a polynomial in the fields it must be separately invariant under the two global $U(1)$ groups generated by T_1 and T_2, respectively. As a gauge group, however, there is a single $(\mathbf{R}, +)$ with a single gauge field. The case of equation (9) is rather unusual and amounts to having some particles with rational charges and others with irrational charges with respect to the same conserved current. Such a situation can never occur if e. g. one has grand unification in mind. The reason is that then G would be a subgroup of a simple gauge group and the elements of the centre of $\mathcal{L}$ would be part of the Cartan subalgebra of the grand unification group. In an irreducible representation of a simple (or semisimple) group the co-roots of simple roots are represented by diagonal matrices with integer entries [1, 3]. Thus with the Gram–Schmidt orthogonalization procedure applied to these generators one can construct an orthonormal basis in the representation of the Cartan subalgebra with rational entries apart from normalization factors. That is, all charges occurring in a matrix Y'_a have rational ratios; hence, if the normalization factor is irrational, it can be put into the gauge coupling constant at energies below the grand unification scale where G is the relevant gauge group.

Finally we would like to mention that we could have made a short-cut in our proof by using the following theorem on Lie algebras [2]: If a Lie algebra of operators on a finite-dimensional vector space is completely reducible then it is the direct sum of a semisimple ideal and its centre. In addition, all elements of the centre are diagonalizable [2]. This theorem is applicable in our case since a unitary representation is completely reducible. It would have summarized most of the arguments of steps 3 to 5. However, we think it would have made the discussion less transparent.

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